Paper Solution AL/CD- 401, Nov 2023

Introduction to Discrete Structure and Linear Algebra.

Sol 1(a)(i) let $x \in A-(B\cap C)$ then

$\Leftrightarrow x \in A$ and $x \notin (B\cap C)$

$\Leftrightarrow x \in A$ and $(x \notin B$ or $x \notin C)$

$\Leftrightarrow (x \in A$ and $x \notin B)$ or $(x \in A$ and $x \notin C)$

$\Leftrightarrow x \in (A-B) \cup x \in (A-C)$

$\Leftrightarrow x \in (A-B)\cup(A-C)$

Hence $A-(B\cap C) = (A-B)\cup(A-C)$

(ii) let (x,y) be any arbitrary element of $A\times(B\cap C)$

then $(x,y) \in A\times(B\cap C) \Leftrightarrow x\in A, y \in (B\cap C)$

$\Leftrightarrow x \in A, (y\in B$ and $y \in C)$

$\Leftrightarrow (x\in A, y\in B)$ and $(x \in A, y \in C)$

$\Leftrightarrow (x,y) \in (A\times B)$ and $(x,y) \in A\times C$

$\Leftrightarrow (x,y) \in (A\times B)\cap(A\times C)$

Hence $A\times(B\cap C) = (A\times B)\cap(A\times C)$

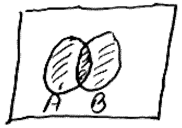
Sol (b) let $x \in (A\cup B)'$ then

$x \in (A\cup B)' \Leftrightarrow x \notin (A\cup B)$

$\Leftrightarrow x \notin A$ ~~and~~ $x \notin B$

$\Leftrightarrow x \in A'$ ~~or~~ $x \in B'$

$\Leftrightarrow x \in (A' \cap B')$



$A\cup B$

Hence $(A\cup B)' = (A' \cap B')$

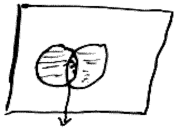
$(A\cap B)' = A'\cup B'$

let $x \in (A\cap B)' \Leftrightarrow x \notin (A\cap B)$

$\Leftrightarrow x \notin A$ or $x \notin B$

$\Leftrightarrow x \in A'$ or $x \in B'$

$\Leftrightarrow x \in A'\cup B'$



$A\cap B$

Hence $(A\cap B)' = A'\cup B'$

Sol 2(a) If $x-y$ is divisible by m Show that this defines an equivalence relation on Z we must relation are reflexive, Symmetric and transitive

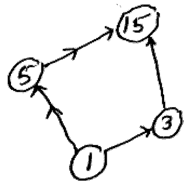
(i) For any x is Z we have $x \equiv x \pmod m$ $x-x=0$, is divisible by m Hence the relation is relflexive

(ii) let $x \equiv y \pmod m$ So $x-y$ is divisible by m then $-(x-y) = (y-x)$ is also divisible by m So $y \equiv x \pmod m$ is also divisible by m Hence the relation is symmetric

(iii) The Sum $(x-y)+(y-z) = (x-z)$ let $x \equiv y \pmod m$ and $y \equiv z \pmod m$ is also divisible by m and $x \equiv z \pmod m$ is also divisible by m the relation is transitive

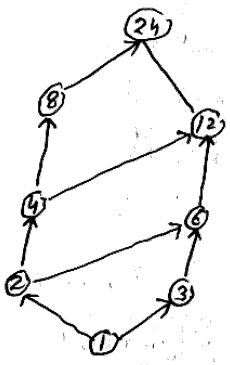
Hence the relation is an equivalance relation

Sol 2(b) let $D_{15} = \{1, 3, 5, 15\}$



D_{15}

$D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$



D_{24}

Sol 3(a) $a * b = a + b - ab.$

G_1:- Closure Property let $a, b, \in \varphi$

$a * b = a + b - ab$ Hence φ is closed w.r.t. operation.

G_2:- Associative let, $a, b, c, \in \varphi$ then we have

$a * (b * c) = a * (b + c - bc)$

$= a + b + c - ab - bc - ca + abc$

$(a * b) * c = (a + b - ab) * c$

$= a + b + c - ab - bc - ca + abc$

Hence $a * (b * c) = (a * b) * c$

$(\varphi\ *)$ is a Semi group

G_3:- Identity let $a \in \varphi$ and e identity element.

$a * e = e * a = a \quad \forall\ a \in \varphi$

$a + e - ae = a \Rightarrow e - ae = 0$

$\Rightarrow e(1 - a) = 0$

$\Rightarrow e = 0, \quad a \neq 1$

So 0 is the identity element.

G_4:- Inverse let $a \in \varphi$, $\exists\ a^{-1} \in \varphi$

$a * a^{-1} = e = a^{-1} * a$

$a + a^{-1} - a a^{-1} = 0$ let, $e = 0$

$a = a^{-1}(a - 1) \Rightarrow a^{-1} = \frac{a}{a-1} \in \varphi$

G_5 Commutative let $a, b \in \varphi$

$a * b = a + b - ab = b + a - b.a = b * a$

thus φ is Commutative

Sol 3(b) Ring:- The ring structure $(R, +, \cdot)$ be defined by following Condition are Satisfied

(1) $R_1 \to (R\ +)$ is an abelian if $a, b, c \in R$

R_{11} (1) Closure law, $a, b\ c, \in R \Rightarrow a + b \in R\ \forall\ a, b \in R$

R_{12} (2) Associative law, $a, b, c \in R$

$(a + b) + c = a + (b + c)\ \forall\ a, b, c \in R$

R_{13} Identity $\forall\ a \in R\ \exists\ 0 \in R$

$a + 0 = 0 + a = a \quad \forall\ a \in R$

R_{14}:- Inverse, $a \in R$ $\exists$ $-a \in R$

$$a + (-a) = (-a) + a = 0$$

R_{15} Commutative law $a, b \in R$

$$a + b = b + a \quad \forall\ a, b \in R$$

R_2 $(R, \cdot)$ is Semi Group.

R_{21}:- Closure law ;- $a, b \in R \Rightarrow a.b \in R$ $\forall\ a.b \in R$

R_{22} Associative law;- $a, b, c \in R$

$$(a.b)c = a.(bc), \quad \forall\ a, b, c \in R$$

R_3:- Distribution law:-

R_{31}:- Left distribution law.

$$a.(b+c) = a.b + a.c, \quad \forall\ a, b, c \in R$$

R_{32}:- Right distribution law.

$$(b+c)a = b.a + c.a, \quad \forall\ a, b, c \in R$$

Commutative ring:- A Commutative ring is which multiplication is Commutative i.e $a.b = b.a$, $\forall\ a, b \in R$

Homomorphism ring:- A ring homomorphism is a function $f: R \to S$ satisfying $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ that is it is a semi-group.

Sol 4(a) $r^3 - 3r^2 + 3r - 1 = 0$, $a_0 = 0$, $a_3 = 3$, $a_5 = 10$

$$(r-1)^3 = 0$$

$$r = 1, 1, 1,$$

$$a_n = (A_0 + A_1 n + A_2 n^2)(1)^n \quad \because 1^n = 1$$

$$a_n = A_0 + A_1 n + A_2 n^2 \quad \text{—(1)}$$

Eqn (1) put $n = 0$

$$a_0 = A_0 + 0 + 0$$

$$\boxed{0 = A_0}$$

Eqn (1) put $n = 3$, $a_3 = 3$

$$3 = A_0 + A_1(3) + A_2(9)$$

$$3 = 0 + 3A_1 + 9A_2 \quad \text{—(2)}$$

Eqn (1) put $n = 5$

$$a_5 = A_0 + 5A_1 + 25A_2$$

$$10 = 0 + 3A_1 + 25A_2 \quad \text{—(3)}$$

Solving (2) & (3) we get

$$A_1 = -0.5, \quad A_2 = 0.5$$

Eqn (1) put A_0, A_1, A_2 we get

$$a_n = 0 + (-0.5)n + 0.5n^2$$

Sol 4(b)

p	q	r	$(p\vee q)$	$\sim p$	$\sim q$	$\sim r$	$(\sim q\vee\sim r)$	$\sim p\wedge(\sim q\vee\sim r)$ (D)	$(\sim p\wedge\sim q)$ (A)	$(\sim p\wedge\sim r)$ (B)	$A\vee B$ (C)	[illegible]	[illegible] (E)	$(E\vee C)$
T	T	T	T	F	F	F	F	F	F	F	F	T	T	T
T	T	F	T	F	F	T	T	F	F	F	F	T	T	T
T	F	T	T	F	T	F	T	F	F	F	F	T	T	T
T	F	F	T	F	T	T	T	F	F	F	F	T	T	T
F	T	T	T	T	F	F	F	F	F	F	F	T	T	T
F	T	F	T	T	F	T	T	T	F	T	F	F	T	T
F	F	T	T	T	T	F	T	T	T	F	F	F	F	T
F	F	F	F	T	T	T	T	T	T	T	T	F	F	T

Hence Tautology

Sol 5(a):–

(i) $p\wedge(p\Rightarrow q)$ $\qquad\because p\Rightarrow q=(\sim p\vee q)$

$\Rightarrow p\wedge(\sim p\vee q)$

$\Rightarrow (p\wedge\sim p)\vee(p\wedge q)$ $\qquad\because p\wedge\sim p=1$

$\Rightarrow (p\wedge q)$

(ii) $\sim p\Rightarrow[r\wedge(p\Rightarrow q)]$

$\sim p\Rightarrow[r\wedge(\sim p\vee q)]$

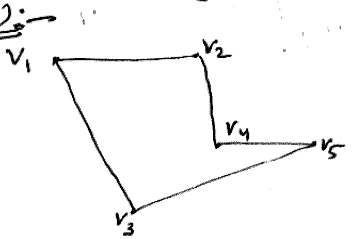
$\sim p\Rightarrow[(r\wedge\sim p)\vee(r\wedge q)]$ dist.

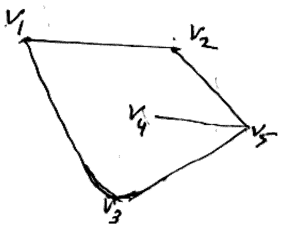
$(p\vee[r\wedge(\sim p\vee q)])$ dist

$(p\vee r)\wedge(p\vee\sim p)\wedge(p\vee q)$

$(p\vee r)\wedge(p\vee q)$

Sol 5(b):–





Sol 6(a):–

The Call Colouring Problem on r Consists in assingning $f(x)$ Colours (frequencies) to each vertex x in V with the Constraint that within every sphere of a given radius r Centred at x, no other points has a Colour in Common with x.

Let G be a Graph there are several different ways to Colour the graph G. Such that no two adjacent vertices have the Same Colour. It is Called the Proper Colouring is also Called Colouring graph.

Sol 6(b):–

using by Cholesky Decomposition

$25x+15y-5z=35$ → ~~divide by 5~~

$15x+18y+0z=33$ → ~~divide by 3~~

$-5x+0y+11z=6$

$25x+15y-5z=35$

$15x+18y+0z=33$

$-5x+0y+11z=6$

this eqn Can be written $AX=B$

$$\begin{bmatrix}25&15&-5\\15&18&0\\-5&0&11\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}35\\33\\6\end{bmatrix}$$

$$A=\begin{bmatrix}25&15&-5\\15&18&0\\-5&0&11\end{bmatrix},\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix},\quad B=\begin{bmatrix}35\\33\\6\end{bmatrix}$$

$$A=L\cdot L^T$$

$$A = LL^T$$

$$\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Comparing first row $l_{11}^2 = 25, \quad l_{11} = 5$

$l_{11} l_{21} = 15 \Rightarrow 5 l_{21} = 15$

$l_{11} l_{31} = -5 \qquad l_{21} = 3$

$5 l_{31} = -5 \Rightarrow l_{31} = -1$

Comp. Second row

$l_{21} l_{11} = 15, \qquad l_{21}^2 + l_{22}^2 = 18$

$l_{21}(5) = 15 \qquad 9 + l_{22}^2 = 18$

$l_{21} = 3 \qquad l_{22} = 3$

$l_{21} l_{31} + l_{22} l_{32} = 0$

$(3)(-1) + l_{32}(3) = 0 \Rightarrow 3 l_{32} = 3$

third row $\quad l_{31}^2 + l_{32}^2 + l_{33}^2 = 11 \qquad l_{32} = 1$

$1 + 1 + l_{33}^2 = 11 \Rightarrow l_{33}^2 = 9$

$l_{33} = 3$

$$L = \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix} \qquad L^T = \begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$LL^T x = B, \quad LY = B \quad L^T x = Y$

$LY = B$

$$\begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 35 \\ 33 \\ 6 \end{bmatrix}$$

$5y_1 = 35 \Rightarrow y_1 = 7,$

$3y_1 + 3y_2 = 33$

$y_1 + y_2 = 11$

$7 + y_2 = 11$

$y_2 = 4$

$-y_1 + y_2 + 3y_3 = 6$

$-7 + 4 + 3y_3 = 6$

$3y_3 = 6 + 3$

$3y_3 = 9$

$y_3 = 3 \qquad Y = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$

$L^T X = Y$

$$\begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$$

$5x + 3y - z = 7$

$3y + z = 4$

$3z = 3$

$(z = 1)$

$3y + 1 = 4$

$y = 1,$

$5x + 3 - 1 = 7$

$5x + 2 = 7$

$x = 1,$

Hence $\quad x = 1, \ y = 1, \ z = 1,$

Sol 7(a):- Null Hypothesis:- there is no statistical Significance difference between the two variable in the Hypothesis is Called the null Hypothesis.

It is generally assumed here that the Hypothesis is true until any other proof has been brought into the light to deny the Hypothesis let us learn more here with definition symbol, principle types and example in this article

often represented by H_0 (H-zero)

Essentially if the between is much Larger than the within variance the factor is Considered statistically significant. Recall Anova seeks to determine a difference in means at each Level of a factor if the factor Level Impacts the mean then that factor is statistically significant.

Sol 7(b) Number of fertilizer, $k=4$

Total number of observations $n=24$

Hypothesis:- H_0:- All population mean are equal.

H_a:- at least one population mean is different from the other population means.

All the Calculation are described within it.

Source of variation	Sum of Squares SS	degree of freedom (df)	Sum of Square $ms = \frac{SS}{df}$	f-value = $\frac{ms(\text{Fertilizer})}{ms(\text{within})}$
Fertilizers	2940	$k-1 = 4-1 = 3$	$\frac{2940}{3} = 980$	$\frac{980}{163.6} = 5.99$
within Groups	$6212-2940 = 3272$	$n-k = 24-4 = 20$	$\frac{3272}{20} = 163.6$	
Total	6212	$n-1 = 24-1 = 23$		

Decision rule

Reject the null if Calculate value > table value

$5.99 > 3.10$

Null Hypothesis are rejected.

Conclusion:- there is sufficient evidence against the null Hypothesis thus we Can say that fertilizers act differently at a 5% Level of Significance.

Sol 8(a) :- Coin is tossed with Probability

$p = \frac{1}{2}$, $q + p = 1$

$q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$n = 400$, Nos of head $x = 216$

$$Z = \frac{x - \mu}{\sigma} = \frac{x - np}{\sqrt{npq}} = \frac{216 - 400(\frac{1}{2})}{\sqrt{400 \cdot (\frac{1}{2})(\frac{1}{2})}}$$

$$= \frac{216 - 200}{\sqrt{100}}$$

$$Z = \frac{16}{10}$$

$$Z = 1.6,$$

If the Coin biased Probability of Coming heads would 0 or 1 but Probability of heads is $\frac{216}{400}$ heance the Coin is unbiased.

Sol 8(b):- $A = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} -4 & 1 \\ -7 & 4 \end{bmatrix}$

Now $AA^T = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} 65 & -32 \\ -32 & 17 \end{bmatrix}$

find the eigen value for AA^T

$|AA^T - \lambda I| = 0 \Rightarrow \begin{vmatrix} 65-\lambda & -32 \\ -32 & 17-\lambda \end{vmatrix} = 0$

$(65 - \lambda)(17 - \lambda) - (-32)(-32) = 0$

$1105 - 82\lambda + \lambda^2 - 1024 = 0$

$\lambda^2 - 82\lambda + 81 = 0$

$(\lambda - 1)(\lambda - 81) = 0$

$\lambda = 1, 81,$ eigen value.